Viscosity of neutron star matter and r-modes in rotating pulsars

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EEK and D.N. Voskresensky (2015) PRC91, 025805; EPJA50,180
Age-period diagram for pulsars

The ATNF pulsar catalogue

Young pulsars’ periods are much larger than the Kepler limit.

\[ \nu_K \simeq 1.2 \text{ kHz} (M/M_\odot)^{1/2} (10 \text{ km}/R)^{3/2} \]

\[ \tau_{s.d.} = P/(2\dot{P}) \]

recycled pulsars

stability of old pulsars?
R-modes in rotating neutron star

Consider a neutron star with a radius $R$ rotating with a rotation frequency $\Omega$ and a perturbation in the form

$$\delta v(\mathbf{r}, t) = a(t) R \Omega \left(\frac{r}{R}\right)^l \mathbf{r} \times \nabla Y_{lm} e^{i\omega t - \tau t / \tau}$$

$$\omega = -\frac{\Omega (l-1)(l+2)}{(l+1)}$$

analogous to Rossby waves in oceans and atmosphere

1998 Andersson and Friedman, Morsink showed that in NS these modes are unstable for $a << 1$ amplitude changes

$$\dot{a}(t) \approx -\frac{a}{\tau}$$

$$\frac{1}{\tau} = -\frac{1}{\tau_G} < 0$$

gravitational radiation drives r-mode unstable

(most unstable for $l=m=2$)

talks by Kostas Kokkotas week 2 and Nils Anderson week 3
If the r-modes are undamped, the star would lose its angular moment on the time scale of $\tau_G$, because of an enhanced emission of gravitation waves.

$$\frac{1}{\tau_G} = \frac{1}{15.6 \, \text{s}} \frac{R_6}{\Omega_4} \frac{\rho_c}{\rho_0}$$

[Lindblom, Owen, Morsnik, PRL80 (98) 4843]

\[
\frac{1}{\tau} = -\frac{1}{\tau_G} + \frac{1}{\tau_V}
\]

**Viscosity**

**Euler equation:**

$$\rho \frac{\partial_i v_i}{\partial t} + \nabla \cdot \mathbf{v} = -\nabla p + \partial_k \sigma_{ki}$$

$$\sigma_{ik} = \eta \left( \nabla_k v_i + \nabla_i v_k - \frac{2}{3} \delta_{ik} \nabla \cdot \mathbf{v} \right) + \zeta \delta_{ik} \nabla \cdot \mathbf{v}$$

**Shear viscosity**

- Dissipation when there is a velocity gradient

**Bulk viscosity**

- Dissipation after uniform volume change

Maxwell (1860) kinetic theory calculations

$$\eta \sim \rho \nu_{\text{rms}} l$$

Units: \(\frac{g}{\text{cm} \cdot \text{s}} = \text{Poise}\)

$$\eta_{\text{Air}} \approx 1.8 \cdot 10^{-4} \text{Poise} \quad \eta_{\text{water}} \approx 1.0 \cdot 10^{-2} \text{Poise}$$
R-modes stability

R-mode is unstable if the time

\[ \frac{1}{\tau} = -\frac{1}{\tau_G} + \frac{1}{\tau_\eta} + \frac{1}{\tau_\zeta} > 0 \]

gravitational time scale

\[ \tau_G^{-1} = \frac{4096}{164025} G \Omega^6 R^7 \langle \rho \rangle_6 \approx \frac{6.43 \cdot 10^{-2}}{s} R_6^7 \Omega_4^6 \rho_{\text{cen}} \frac{\rho_0}{\rho} \]

damping rate due to the shear viscosity

\[ \tau_\eta^{-1} = \frac{5}{R^2} \frac{\langle \eta \rangle_4}{\langle \rho \rangle_6} \approx \frac{5.98 \cdot 10^{-5}}{s} \frac{\langle \eta_{20} \rangle_4}{R_6^2 \rho_{\text{cen}} / \rho_0} \]

damping rate due to the bulk viscosity

\[ R_6 = R / 10^6 \text{ cm} \]
\[ \Omega_4 = \Omega / 10^4 \text{ s} \]
\[ \eta_{20} = \eta / (10^{20} \text{ g cm s}^{-1}) \]
\[ \zeta_{20} = \zeta / (10^{20} \text{ g cm s}^{-1}) \]

\[ \tau_\zeta^{-1} = \frac{4\pi}{690} \left( \frac{\Omega^2}{\pi G \rho} \right)^2 \frac{\langle \zeta (1 + 0.86(r/R)^2) \rangle_8}{R^2 \langle \rho \rangle_6} \approx \frac{2.20 \cdot 10^{-7}}{s} R_6^4 \Omega_4^4 \frac{\langle \zeta_{20} [1 + 0.86(r/R)^2] \rangle_8}{(M/M_\odot)^2 (\rho_{\text{cen}} / \rho_0)} \]

\[ \langle \ldots \rangle_n = \frac{1}{R^{n+1}} \int_0^R (\ldots) r^n \text{ dr} \]

Profile averages

[Lindblom, Owen, Morsnik PRL 80 (98) 4843
Owen, Lindblom, Cutler, Schutz, Vecchio, Andersen PRD58 (98) 084020]
**Shear viscosity**: collisional viscosity: 
*lepton, nucleon, phonon, neutrino contributions*

**Bulk viscosity**: collisional viscosity 
“soft-mode” viscosity (Leontovich-Mandelstam)

**Inputs**: EoS, density profiles in NS, pairing gaps, NN-interaction

- **neutron star structure**
  - mass and radius
  - density profiles

  - for $n > 0.6n_0$ HDD EoS
  - for $n \leq 0.6n_0$ Friedman-Pandharipande-Skyrme EoS

\[
\frac{\rho(r)}{\rho(0)} \simeq \frac{n(\rho)}{n(0)} \simeq f\left(\frac{r}{R}\right) = 1 - \left(\frac{r}{R}\right)^2
\]

\[
\frac{X_p(r)}{X_p(0)} \simeq \chi\left(\frac{r}{R}\right) = \left[1 - \left(\frac{r}{R}\right)^2\right]^{3/5}
\]
Profiles of critical temperatures for the singlet neutron and proton pairing

We ignore $^3P_2$ neutron pairing

The heavier a NS is the smaller is the size of the region with nucleon pairing
shear viscosity

$$\eta = \frac{1}{15T} \sum_a \int \frac{d^3p}{(2\pi)^3} \tau_a (v^2 p^2) n_a(p) (1 \pm n_a(p))$$

degenerated fermions $T \ll T_F$

$$\eta = \frac{1}{5} \sum_a n_a p_{F,a} \tau_a$$

collision time of the particle $a$

Fermi liquid result $\tau_a \propto \frac{1}{T^2}$

bulk viscosity

$$\zeta = \frac{1}{9T} \int \frac{d^3p}{(2\pi)^3} \tau (v^2 - 3v_s^2)^2 m^2 n_p (1 \pm n_p)$$
**Lepton shear viscosity**  
Lepton shear viscosity = electron + muon contribution  
\[ \eta_{e/\mu} = \eta_e + \eta_\mu \]

low T, Fermi liquid results:  
\[ \eta_l = \frac{1}{5} n_l p_{F,l} \tau_l \]

Lepton collision time \( \tau_l \) is determined by lepton-lepton and lepton-proton collisions

Flowers and Itoh:  
\[ \eta_{e/\mu}^{(FI)} \approx \eta_e^{(FI)} = 4.2 \cdot 10^{17} \left[ \frac{g}{\text{cm} \cdot \text{s}} \right] \left( \frac{\rho}{\rho_0} \right)^2 T_9^{-2} \]

**Important role of the phonon modification (plasmon exchange)**


**leading terms for small T**

\[ \eta_e = 1.82 \cdot 10^{19} \left[ \frac{g}{\text{cm} \cdot \text{s}} \right] \left( \frac{n_p}{n_0} \right)^{\frac{14}{9}} \left( \frac{n_e}{n_p} \right)^2 \frac{T_9^{\frac{5}{3}}}{(1 + r)^{\frac{2}{3}}} \]

\[ r = \left( \frac{p_{F,e}^2 + p_{F,\mu}^2}{p_{F,p}^2} \right) \]

\[ \eta_\mu = \left( \frac{n_\mu}{n_e} \right)^{\frac{5}{3}} \eta_e \]

muon contributions are important
Lepton shear viscosity vs. neutron star mass

Effects of proton pairing on lepton shear viscosity

\[ S_{\eta,e/\mu} = \frac{\langle \eta_{e/\mu}^{(s)} \rangle}{\langle \eta_{e/\mu,20} \rangle} \]

Nucleon shear viscosity

**Fermi liquid result**

\[
\eta_n = \frac{3n_n p_{F,n}^2 m^2_N}{80m_n^* T^2 S_{nn}}
\]

**effective NN cross section**

\[
S_{nn} = \frac{m^2_N}{16\pi^2} \int_0^1 dx' \int_0^1 \frac{\sqrt{1-x'^2}}{dx} 12x^2 x'^2 Q_{nn}(q, q') \sqrt{1-x^2-x'^2}
\]

**FOPE:**

\[
S_{nn}^{\text{FOPE}} \sim \frac{3 m_n^2 f_{\pi NN}^4}{40\pi} \sim \frac{1.1}{m^2_\pi}
\]

**MOPE:**

\[
S_{nn}^{\text{MOPE}} = K S_{nn}^{\text{FOPE}}
\]

\[
K = \frac{30\pi \Gamma^4 p_{F,n}^3}{128\gamma \tilde{\omega}^3} \left[ 1 + \frac{2}{3\gamma p_{F,n}} \tilde{\omega} \right]
\]

**modification factor**

\[
K(n = n_0) \simeq 0.3 \quad [\text{Bacca et al, PRC80}]
\]

\[
K(n = 2.6n_0) \simeq 1
\]

\[
K(n \geq 3n_0) \simeq 2
\]

\[
\eta_n \ll \eta_l
\]
Phonon shear viscosity

We consider the interaction of the phonon (Anderson-Bogoliubov) mode with neutrons

\[
\eta_{ph} = \int \frac{d^3q}{(2\pi)^3} \frac{\tau_{ph}}{15T} \frac{(sv_{F,n})^2 q^2}{(e^{sv_{F,n}q/T} - 1)(1 - e^{-sv_{F,n}q/T})} = \frac{2\pi^2}{25} \frac{T^4}{v_{F,n}^3} \bar{\tau}_{ph} \quad s = 1/\sqrt{3}
\]

From the Larkin-Migdal equation for anomalous vertex \[\text{[E.K. Voskresensky, PRC77, PRC81]}\]

\[
\tilde{\tau}_{V,0} = \frac{-2 \Delta \omega}{\omega^2 - \frac{1}{3} v_{F,n}^2 q^2 - i\omega \gamma_{ph}(\omega, q)} \tilde{\tau}_{V,0}^{\omega}
\]

\[
\gamma_{ph}(q) = 1/\tau_{ph} \approx \frac{2\pi}{3} v_{F,n} q e^{-\sqrt{\frac{3}{2}} \frac{\Delta}{T}}
\]

The phonon lifetime \(\bar{\tau}_{ph} \approx 5.9 \cdot 10^{-22} e^{\sqrt{\frac{3}{2}} \frac{\Delta}{T}} \frac{s}{T_0}\) must be smaller than the balistic time

\[
\tau_{bal} \sim \frac{1}{s} \frac{km}{v_F} \approx 1.6 \cdot 10^{-5} s \left( \frac{n_0}{n} \right)^{1/3} \frac{m^*_n}{m_N}
\]

size of the region of the neutron pairing
\[ \eta_{ph} \simeq 2.1 \cdot 10^{23} \left[ \frac{g}{cm \cdot s} \right] \left( \frac{n_0}{n} \right) \left( \frac{m_n^*}{m_N} \right)^3 T_9^4 \frac{\min\{\tau_{ph}, \tau_{bal}\}}{s} \]

\[ \tau_{ph} \simeq 5.9 \cdot 10^{-22} e \sqrt{\frac{3}{2}} \frac{\Delta}{T_9} s \]

\[ \tau_{bal} \simeq 1.6 \cdot 10^{-5} s \left( \frac{n_0}{n} \right)^{\frac{1}{3}} \frac{m_n^*}{m_N} \]

\( \eta_{ph} \) strongly depends on the pairing gap, contributes at temperatures slightly below \( T_c \).
Neutrino shear viscosity

With the temperature increase the neutrino mean free path decreases and for sufficiently high temperatures neutrinos become trapped inside the neutron star interior.

\[
\eta_\nu = 2 \int \frac{2d^3q}{(2\pi)^3} \frac{\tau_\nu}{15T} \frac{v_\nu^2 q^2}{(e^{v_\nu q/T} + 1)(1 + e^{-v_\nu q/T})} = \frac{7 \pi^2}{225 v_\nu^2} T^4 \tau_\nu
\]

\[
\mu_\nu \approx 0 \quad v_\nu = c
\]

Neutrino mean free path is determined by inverse MMU and PU processes

\[
\tau_\nu \approx \frac{8.7 \text{ s}}{T_9^4 F_{\text{MMU}}(n)} \left( \frac{m_N}{m_N^*} \right)^4 \frac{(n_0/n_e)^{1/3}}{1 + \chi_{\text{PU}}(n,T)}
\]

\[
\eta_\nu \approx \frac{3.08 \cdot 10^{22}}{1 + \chi_{\text{PU}}(n,T)} \left[ \frac{\text{g/cm s}}{1} \right] \left( \frac{n_0}{n_p} \right)^{1/3} \left( \frac{m_N}{m_N^*} \right)^4 F_{\text{MMU}}(n)
\]

Contributes only in regions where neutrinos are trapped

\[
\eta_\nu^{\text{opac}}(r,T) = \eta_\nu(n(r)) \theta(r_{\text{opac}} - r)
\]

Opacity radius is determined as

\[
v_\nu \tau_\nu(n(r_{\text{opac}}),T) = R - r_{\text{opac}}
\]
Lepton shear viscosity

\[ \langle \eta \rangle_4 \ [g/(cm \ s)] \]

\[ 10^{15} - 10^{22} \]

\[ T \ [K] \]

\[ 10^7 - 10^{10} \]

- Leptons
- Neutrinos

\[ M/M_\odot \]

- 1.0
- 1.5
- 2.05

MMU

MMU+PU
Bulk viscosity

\[ \zeta_{\text{coll}} = \frac{m_N^*}{162\pi^2} \frac{3}{n_0} \tau T^4 \left[ \frac{n_0}{n} \right]^{1/3} F_0^2 \]


\( F_0 \) is the zeroth harmonics of the dimensionless scalar Landau-Migdal parameter, \( F_0 \sim 1 \)

\( \tau \sim \frac{m_\pi^2}{(m_N^* T^2)} \) nucleon relaxation time;

\[ \zeta_{\text{coll}} \sim 90 \left[ \frac{g}{\text{cm} \cdot \text{s}} \right] T^2 \left[ \frac{n_0}{n} \right]^{1/3} F_0^2 \]

small contribution
Energy dissipation of the mode: \( \dot{E}_{\text{mode}} = P \dot{V} - \epsilon_\nu \) \( \times \) is neutrino emissivity

Energy of the mode decreases if the pressure depends on an **order parameter**, which variation is **delayed** with respect to the variation of the density [Mandelstam, Leontovich, ZhETF 7 (1937) 438]

in neutron stars

**order parameter** is \( X_l = n_l/n \)  **lepton concentration** \( \delta \mu_l = \mu_n - \mu_p - \mu_l \neq 0 \)

\[
\dot{X}_l = -\frac{\delta X_l}{\tau_{X,l}} + n \frac{\partial \delta \mu_l}{\partial n} \delta n(t)
\]

relaxation time

soft mode contribution

\[
\zeta_{\text{s.m.}} = n \frac{\langle P(n + \delta n(t), X_l + \delta X_l(t)) \delta \dot{n}(t) \rangle_P}{\langle (\delta \dot{n}(t))^2 \rangle_P}
\]

average over the perturbation period

\[ \zeta_{s.m.} \approx -\frac{\partial P}{\partial X_l} \frac{dX_l}{dn} \frac{n\tau_{X,l}}{1 + \omega^2 \tau_{X,l}^2} \approx -\frac{\partial P}{\partial X_l} \frac{dX_l}{dn} \frac{n}{\omega^2 \tau_{X,l}} \quad \text{for} \quad \omega \tau_{X,l} \gg 1 \]

\[ \frac{1}{\tau_{X,l}} = \sum_r R^{[r]} = \sum_r \frac{1}{\tau_{X,l}^{[r]}} \]  

\[ \zeta_{s.m.} \approx \sum_r \zeta^{[r]}_{s.m.} \]  

direct Urca (DU); \quad \zeta^{[DU]}_{s.m.} \propto T^5

modified Urca (MU) \[ \text{[calculated with FOPE]} \] \quad \zeta^{[MU]}_{s.m.} \propto T^7

or

medium MU (MMU) \[ \text{[calculated with MOPE]} \] \quad \zeta^{[MMU]}_{s.m.} \propto F_{MMU}(n)T^7

pion condensate Urca (PU) \quad \zeta^{[PU]}_{s.m.} \propto T^5

\[ \langle \zeta^{(r)}_{s.m.} (1 + 0.86 r^2 / R^2) \rangle \]

\[ T = 10^9 K, \quad \omega = 4/3 \times 10^4 \text{Hz} \]
Shear and bulk viscosities. Results

\[ [g/cm \ s] \]

\[ T [K] \]

\[ M/M_\odot \]

- Red line: 1.0
- Red dashed line: 1.5
- Red dotted line: 1.6
- Red dash-dotted line: 1.8
- Red dot-dash line: 1.9
- Red dot-dot-dash line: 2.05

\[ \langle \eta \rangle_4 \]

\[ \langle \zeta(...) \rangle_8 \]

\[ \langle \eta \rangle_4 \]
R-mode stability window

\[ \tau^{-1}_G(\nu_c) = \tau^{-1}_\eta(\nu_c) + \tau^{-1}_\zeta(\nu_c) \quad \rightarrow \quad \nu_c = \nu_c(T) \]

\[ \nu = \Omega/2\pi \]

![Graph illustrating R-mode stability window with various curves representing different mass ranges and types of energy dissipation. The graph shows the relationship between \( \nu_c \) and \( T \) in kiloKarnel temperature, with shear and bulk regions indicated.]
Young pulsars

1. star is born hot and rapidly rotating (point A)

2. cooling time (heat transport!) $\gg$ spin-down time

$$t_{\text{spin-down}} \sim \frac{100 \text{s}}{a_{\text{max}}^2 \nu_3^6}$$

for max. r-mode amplitude $a_{\text{max}} \sim 1$

3. star moves along line AB because of r-modes

4. line BC, cooling and magnetic breaking

Minimum point B must be above 62 Hz (PSRJ0537-6910)
Minimum of the stability window

Mass is too close to the limiting one
Rotation of LMXB pulsars cannot be explained. Shear viscosity is too small.

**Alternative mechanisms**

- differential drift in magnetic field  
  [Rezzolla, Lamb, Shapiro]
- weak reactions with hyperons + hyperon pairing  
  [Jones; Nayyar Owen]
- core-crust coupling  
  [Bildsten, Ushomirsky, Levin]
- saturation of r-mode amplitude at small values  
  [Arras, Bondaresku, Wasserman]
- non-linear decay of r-modes  
  [Kastaun]
- coupling to more stable modes  
  [Gusakov, Chugunov, Kantor]
- vortex flux-tube interactions  
  [Haskell, Glampedakis, Andersson]
Conclusions

In-medium modifications of modes is important in the NS core:
- moderate increase nucleon shear viscosity for small NS masses
- partial trapping of neutrinos in the star core at $T \approx 2 \times 10^9 - 10^{10}$ K
- large neutrino shear viscosity for $T > 2 \times 10^9$
- strong increase of the bulk viscosity

Phonon shear viscosity (due to the phonon-neutron interaction)
- could be important if the gap is sufficiently large

Rotation periods of young pulsars can be understood
- if the pion softenning is taken into account

Shear viscosity at $T \approx 10^8$ K is too small to explain recycled pulsars in LMXB
- differential rotation of the star → increase of the critical frequency
Outer part of the core does not rotate. Crust rotates \[
\frac{\Omega_{\text{fin}}(r)}{\Omega} \approx \theta(r_c - r)
\]

Minimal \(r_c\) is determined by the size of the proton paring zone.
Soft bosonic modes in rapidly rotating systems

dispersion laws of the low-lying bosonic excitations

What happens if the medium flows with a velocity \( v > v_c \)?

\[ v_c = \frac{\epsilon(k_0)}{k_0} \]

[Pitaevskii '84; Voskresenky 93; Baym, Pethick 12]

There appears a Bose condensate of excitations, which will carry a part of the momentum and the fluid will move with smaller velocity.

\[ \varphi = \varphi_0 e^{-i\epsilon(k_0)t + ik_0r} \]

\[ \rho v = \rho v_{\text{fin}} + k_0|\varphi_0|^2 \]

\[ \delta E = -(v k - \epsilon(k))|\varphi_0|^2 + \frac{1}{2}\Lambda|\varphi_0|^4 \]

differential rotation of the star